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Global exponential stability of impulsive differential equations with any time delays

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ABSTRACT

The main objective of this letter is to further investigate the global exponential stability of a class of general impulsive retarded functional differential equations. Several new criteria on global exponential stability are analytically established based on Lyapunov function methods combined with Razumikhin techniques. The obtained results extend and generalize some results existing in the literature. An example, along with computer simulations, is included to illustrate the results.

1. Introduction

In recent years, the issues of stability in impulsive differential equations with time delays have attracted increasing interest in both theoretical research and practical applications (see [1–8] for references). In particular, special attention has been focused on exponential stability of delay differential equations because it has played an important role in many areas such as designs and applications of neural networks, synchronization of coupled oscillators and consensus problems of networked control systems (see [9–11] and the relevant references therein).

There are several research works which appeared in the literature on exponential stability of impulsive delay differential equations [1–3]. A few main results for some special classes of impulsive delay differential equations have been obtained, see [4–6] for example. In [1], Wang and Liu proved several criteria on exponential stability for a class of general impulsive delay differential equations by utilizing Lyapunov function methods combined with Razumikhin techniques. *However, these criteria are only valid for some specific small delays* due to the restrictive requirement that the time delays are smaller than the length of each impulsive interval, so they are generally inapplicable to some practical applications. The main objective of this letter is to investigate the exponential stability of such impulsive delay differential equations *for any time delays*. As a result, several new criteria on global exponential stability are analytically derived, which are natural extension and generalization of the corresponding results existing in the literature. It should also be noticed that our work was inspired in part by the recent work of Wang and Liu in [1], and some arguments from [1] were also employed. However, here the analysis technique we use is different from that used by Wang and Liu in [1], which is fully applicable to deal with global exponential stability of impulsive differential equations for any time delays. Moreover, our result shows that impulses do contribute to global exponential stability of dynamical systems with any time delays even if they are unstable, which can be usually used as an effective control strategy to stabilize the underlying delay dynamical systems in some practical applications.

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2. Main results

Consider the following impulsive retarded differential equation [1]:

$$\begin{cases} \dot{x}(t) = f(t, x_t), & t \neq t_k,\ t \geq t_0 \\ \Delta x(t_k) = I_k(t_k, x_{t_k^-}), & k \in \mathbb{N} \\ x_{t_0} = \phi. \end{cases} \tag{1}$$

As in [1,2], we assume that functions $f, I_k : \mathbb{R}_+ \times PC([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}^n$; $\phi \in PC([-\tau, 0], \mathbb{R}^n)$, satisfy all necessary conditions for the global existence and uniqueness of solutions for all $t \geq t_0$ [12]. The time sequence $\{t_k\}_{k=1}^{+\infty}$ satisfy $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots$, $\lim_{k\to\infty} t_k = \infty$, $\Delta x(t_k) = x(t_k) - x(t_k^-)$, and $x_t, x_{t^-} \in PC([-\tau, 0], \mathbb{R}^n)$ are defined by $x_t(s) = x(t+s)$ and $x_{t^-}(s) = x(t^- + s)$ for $-\tau \leq s \leq 0$, respectively. We shall assume $f(t, 0) = I_k(t, 0) = 0$ for all $t \in \mathbb{R}_+$ and $k \in \mathbb{N}$ so that system (1) admits the trivial solution. Given a constant $\tau > 0$, we equip the linear space $PC([-\tau, 0], \mathbb{R}^n)$ with the norm $\|\cdot\|_\tau = \sup_{-\tau\leq s\leq 0} \|\psi(s)\|$. Denote $x(t) = x(t, t_0, \phi)$ the solution of (1) such that $x_{t_0} = \phi$. We further assume that all the solutions $x(t)$ of (1) are continuous except at t_k, $k \in \mathbb{N}$, at which $x(t)$ is right continuous, i.e., $x(t_k^+) = x(t_k)$, $k \in \mathbb{N}$. For some definitions with respect to a class of ν_0 functions and its upper right-hand derivative, and global exponential stability of impulsive retarded differential equations, we refer to [1,2], and hence are omitted from this paper due to the limitation of space.

In the following, we shall establish several criteria on global exponential stability for impulsive differential equation with any time delays. Our result shows that impulses play an important role in making a retarded dynamical system globally exponentially stable even if it may be unstable itself.

Theorem 2.1. *Assume that there exist a function $V \in \nu_0$ and several positive constants $p, c, c_1, c_2, \sigma, \lambda > 0$, $\gamma \geq 1$ and $\sigma - \lambda \geq c$ such that*

(i) $c_1\|x\|^p \leq V(t, x) \leq c_2\|x\|^p$, *for any $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$;*
(ii) $D^+V(t, \varphi(0)) \leq cV(t, \varphi(0))$, *for all $t \in [t_{k-1}, t_k)$, $k \in \mathbb{N}$, whenever $qV(t, \varphi(0)) \geq V(t+s, \varphi(s))$ for $s \in [-\tau, 0]$, where $q \geq \gamma e^{\lambda\tau}$ is a constant;*
(iii) $V(t_k, \varphi(0) + I_k(t_k, \varphi)) \leq d_k V(t_k^-, \varphi(0))$, *where $0 < d_{k-1} \leq 1$, $\forall k \in \mathbb{N}$, are constants;*
(iv) $\gamma \geq 1/d_{k-1}$ *and* $\ln d_{k-1} < -(\sigma+\lambda)(t_k - t_{k-1})$, $k \in \mathbb{N}$.

Then the zero solution of the impulsive retarded differential equation (1) is globally exponentially stable with convergence rate λ/p for any time delays $\tau \in (0, \infty)$.

Proof. Let $x(t) = x(t, t_0, \phi)$ be any solution of the impulsive system (1) with the initial condition $x_{t_0} = \phi$, and $v(t) = V(t, x)$.

We shall show that

$$v(t) \leq M\|\phi\|_\tau^p e^{-\lambda(t-t_0)}, \quad t \in [t_{k-1}, t_k),\ k \in \mathbb{N}. \tag{2}$$

Let $\gamma \geq \sup_{k\in\mathbb{N}}\{\frac{1}{d_{k-1}}\}$. From condition (iv), we can choose a positive constant $M > 0$ such that

$$0 < c_2 e^{(\sigma+\lambda)(t_1-t_0)} \leq M \leq c_2\gamma e^{\lambda\tau-(\sigma+\lambda)(t_1-t_0)} e^{(\sigma+\lambda)(t_1-t_0)}. \tag{3}$$

It then follows that

$$0 < c_2\|\phi\|_\tau^p < c_2\|\phi\|_\tau^p e^{\sigma(t_1-t_0)} \leq M\|\phi\|_\tau^p e^{-\lambda(t_1-t_0)}. \tag{4}$$

We first prove that

$$v(t) \leq M\|\phi\|_\tau^p e^{-\lambda(t-t_0)}, \quad t \in [t_0, t_1). \tag{5}$$

To do this, we only need to prove that

$$v(t) \leq M\|\phi\|_\tau^p e^{-\lambda(t_1-t_0)}, \quad t \in [t_0, t_1). \tag{6}$$

If (6) is not true, then by (4) there exists $\bar{t} \in (t_0, t_1)$ such that

$$v(\bar{t}) > M\|\phi\|_\tau^p e^{-\lambda(t_1-t_0)} \geq c_2\|\phi\|_\tau^p e^{\sigma(t_1-t_0)} > c_2\|\phi\|_\tau^p \geq v(t_0+s), \quad s \in [-\tau, 0],$$

which implies that there exists $t^* \in (t_0, \bar{t})$ such that

$$v(t^*) = M\|\phi\|_\tau^p e^{-\lambda(t_1-t_0)}, \quad \text{and} \quad v(t) \leq v(t^*), \quad t \in [t_0-\tau, t^*], \tag{7}$$

and there exists $t^{**} \in [t_0, t^*)$ such that

$$v(t^{**}) = c_2\|\phi\|_\tau^p, \quad \text{and} \quad v(t^{**}) \leq v(t) \leq v(t^*), \quad t \in [t^{**}, t^*]. \tag{8}$$

Hence, for any $s \in [-\tau, 0]$, by (3) and (8), we get

$$\begin{aligned} v(t+s) &\le M\|\phi\|_\tau^p e^{-\lambda(t_1-t_0)} \le c_2\gamma e^{\lambda\tau-(\sigma+\lambda)(t_1-t_0)} e^{(\sigma+\lambda)(t_1-t_0)}\|\phi\|_\tau^p e^{-\lambda(t_1-t_0)} \\ &\le \gamma e^{\lambda\tau} c_2\|\phi\|_\tau^p = \gamma e^{\lambda\tau} v(t^{**}) \le q v(t^{**}) \le q v(t), \quad t \in [t^{**}, t^*], \end{aligned} \tag{9}$$

and thus by (9) and condition (ii), for $t \in [t^{**}, t^*]$, we get $D^+v(t) \le cv(t) \le (\sigma-\lambda)v(t)$. It follows from (3), (7) and (8) that

$$\begin{aligned} v(t^*) &\le v(t^{**}) e^{(\sigma-\lambda)(t^*-t^{**})} < c_2\|\phi\|_\tau^p e^{(\sigma-\lambda)(t_1-t_0)} < c_2\|\phi\|_\tau^p e^{\sigma(t_1-t_0)} \\ &= c_2\|\phi\|_\tau^p e^{(\sigma+\lambda)(t_1-t_0)} e^{-\lambda(t_1-t_0)} \le M\|\phi\|_\tau^p e^{-\lambda(t_1-t_0)} = v(t^*), \end{aligned}$$

which is a contradiction. Hence (5) holds and then (2) is true for $k = 1$.

Now we assume that (2) holds for $k = 1, 2, \ldots, m$ ($m \in \mathbb{N}$, $m \ge 1$), i.e.,

$$v(t) \le M\|\phi\|_\tau^p e^{-\lambda(t-t_0)}, \quad t \in [t_{k-1}, t_k),\ k = 1, 2, \ldots, m. \tag{10}$$

Next, we shall show that (2) holds for $k = m+1$, i.e.,

$$v(t) \le M\|\phi\|_\tau^p e^{-\lambda(t-t_0)}, \quad t \in [t_m, t_{m+1}). \tag{11}$$

Suppose (11) is not true. Then we define $\bar{t} = \inf\{t \in [t_m, t_{m+1}) | v(t) > M\|\phi\|_\tau^p e^{-\lambda(t-t_0)}\}$. From conditions (iii), (iv) and (10), we get

$$\begin{aligned} v(t_m) &\le d_m M\|\phi\|_\tau^p e^{-\lambda(t_m-t_0)} = d_m M\|\phi\|_\tau^p e^{\lambda(\bar{t}-t_m)} e^{-\lambda(\bar{t}-t_0)} < d_m e^{\lambda(t_{m+1}-t_m)} M\|\phi\|_\tau^p e^{-\lambda(\bar{t}-t_0)} \\ &< e^{-(\sigma+\lambda)(t_{m+1}-t_m)} e^{\lambda(t_{m+1}-t_m)} M\|\phi\|_\tau^p e^{-\lambda(\bar{t}-t_0)} < M\|\phi\|_\tau^p e^{-\lambda(\bar{t}-t_0)}, \end{aligned} \tag{12}$$

and hence $\bar{t} \ne t_m$. From the continuity of $v(t)$ on the interval $[t_m, t_{m+1})$, we have

$$v(\bar{t}) = M\|\phi\|_\tau^p e^{-\lambda(\bar{t}-t_0)}, \quad \text{and} \quad v(t) \le v(\bar{t}), \quad t \in [t_m, \bar{t}]. \tag{13}$$

From (12), we know that there exists $t^* \in (t_m, \bar{t})$ such that

$$v(t^*) = d_m e^{\lambda(t_{m+1}-t_m)} M\|\phi\|_\tau^p e^{-\lambda(\bar{t}-t_0)}, \quad \text{and} \quad v(t^*) \le v(t) \le v(\bar{t}), \quad t \in [t^*, \bar{t}]. \tag{14}$$

On the other hand, for $t \in [t^*, \bar{t}]$ and $s \in [-\tau, 0]$, either $t+s \in [t_0-\tau, t_m)$ or $t+s \in [t_m, \bar{t}]$. If $t+s \in [t_0-\tau, t_m)$, from (10), we obtain

$$\begin{aligned} v(t+s) &\le M\|\phi\|_\tau^p e^{-\lambda(t-t_0)} e^{-\lambda s} \le M\|\phi\|_\tau^p e^{-\lambda(\bar{t}-t_0)} e^{\lambda(\bar{t}-t)} e^{\lambda\tau} \\ &\le e^{\lambda\tau} e^{\lambda(t_{m+1}-t_m)} M\|\phi\|_\tau^p e^{-\lambda(\bar{t}-t_0)}, \end{aligned} \tag{15}$$

while, if $t+s \in [t_m, \bar{t}]$, from (13), then

$$v(t+s) \le M\|\phi\|_\tau^p e^{-\lambda(\bar{t}-t_0)} \le e^{\lambda\tau} e^{\lambda(t_{m+1}-t_m)} M\|\phi\|_\tau^p e^{-\lambda(\bar{t}-t_0)}. \tag{16}$$

In any case however, (14)–(16) imply that, for any $s \in [-\tau, 0]$, we have

$$v(t+s) \le e^{\lambda\tau} e^{\lambda(t_{m+1}-t_m)} M\|\phi\|_\tau^p e^{-\lambda(\bar{t}-t_0)} \le \gamma e^{\lambda\tau} v(t^*) \le \gamma e^{\lambda\tau} v(t) \le q v(t), \quad t \in [t^*, \bar{t}]. \tag{17}$$

Finally, by (17) and condition (ii), we have $D^+v(t) \le (\sigma-\lambda)v(t)$. Thus, in view of condition (iv), we have

$$\begin{aligned} v(\bar{t}) &\le v(t^*) e^{(\sigma-\lambda)(\bar{t}-t^*)} = d_m e^{\lambda(t_{m+1}-t_m)} M\|\phi\|_\tau^p e^{-\lambda(\bar{t}-t_0)} e^{(\sigma-\lambda)(\bar{t}-t^*)} \\ &< e^{-(\sigma+\lambda)(t_{m+1}-t_m)} e^{\lambda(t_{m+1}-t_m)} M\|\phi\|_\tau^p e^{-\lambda(\bar{t}-t_0)} e^{(\sigma-\lambda)(\bar{t}-t^*)} \\ &= M\|\phi\|_\tau^p e^{-\sigma(t_{m+1}-t_m)} e^{(\sigma-\lambda)(\bar{t}-t^*)} e^{-\lambda(\bar{t}-t_0)} \\ &< M\|\phi\|_\tau^p e^{-\lambda(t_{m+1}-t_m)} e^{-\lambda(\bar{t}-t_0)} < M\|\phi\|_\tau^p e^{-\lambda(\bar{t}-t_0)} = v(\bar{t}), \end{aligned}$$

which is a contradiction. This implies the assumption is not true, and hence (11) holds. Therefore, by some mathematical induction, we obtain (2) holds for any $k \in \mathbb{N}$. Then from condition (i), we have

$$\|x\| \le M^*\|\phi\|_\tau e^{-\frac{\lambda}{p}(t-t_0)}, \quad t \in [t_{k-1}, t_k), k \in \mathbb{N}$$

where $M^* \ge \max\{1, [\frac{M}{c_1}]^{\frac{1}{p}}\}$, which implies that the zero solution of the impulsive system (1) is globally exponentially stable with convergence rate λ/p. The proof is completed. □

Remark 2.1. It seems that the proof of Theorem 2.1 is similar to that of the main result of Wang and Liu (see Theorem3.1 in [1]). However, they are actually different. In fact, a distinct feature of Theorem 2.1 is to remove the restrictive condition that time delays are required to be less than the length of each impulsive interval, i.e., the additional assumption (iv): $\tau \le t_k - t_{k-1} \le \alpha$ in Theorem 3.1–3.2 of [1] is indeed deleted here. Moreover, a controlled parameter σ is introduced to adjust the magnitude of convergence rate λ/p on global exponential stability of impulsive retarded differential equations (1). Therefore, Theorem 2.1 is more practically applicable than those existing in the recent literature (see for example: Theorems 3.1–3.2 of [1] and Theorem 3.1 of [2]) since it can be applied to deal with global exponential stability of impulsive differential equations for any time delays $\tau \in (0, +\infty)$. This point will be further illustrated through the following Example 2.1.

Theorem 2.1 can be used to deal with the following linear impulsive delay differential equations [1]:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bx(t - \tau(t)), & t \neq t_k, t \ge t_0 \\ \Delta x(t) = C_k x(t^-), & t = t_k,\ k \in \mathbb{N} \\ x_{t_0} = \phi, \end{cases} \tag{18}$$

where $t - \tau(t)$ is strictly increasing on $\mathbb{R}_+$ and $0 \le \tau(t) \le \tau$.

Corollary 2.1. *Suppose there exist some constants σ, $\lambda > 0$, and $\gamma \ge 1$ such that*

(i) *for some constant $q \ge \gamma e^{\lambda\tau}$, $\lambda_{\max}(A^T + A + E) + q\|B\|^2 \le \sigma - \lambda$;*
(ii) *$\gamma \ge 1/\|E + C_{k-1}\|$ and $\ln \|E + C_{k-1}\| < -(\sigma + \lambda)(t_k - t_{k-1})$, where $C_0 = E$, $k \in \mathbb{N}$.*

Then system (18) is globally exponentially stable and its convergence rate is $\lambda/2$.

Proof. It follows from Theorem 2.1 by selecting Lyapunov function $V(x) = \|x\|^2$. □

Example 2.1. Consider the following linear impulsive retarded dynamical system [1]:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bx\left(t - \dfrac{2}{5}(1 + e^{-t})\right), & t \neq t_k,\ t \ge t_0 \\ \Delta x(t) = C_k x(t^-), & t = t_k,\ k \in \mathbb{N} \\ x_{t_0} = \phi, \end{cases} \tag{19}$$

where $C_k = \mathrm{diag}(-0.5, -0.8, -0.4)$, and

$$A = \begin{pmatrix} 0.1 & 0.2 & -0.1 \\ 0.2 & 0.15 & 0.3 \\ 0 & 0.24 & 0.1 \end{pmatrix}, \qquad B = \begin{pmatrix} -0.12 & 0.02 & 0 \\ 0.12 & -0.2 & 0.05 \\ 0 & 0.14 & -0.1 \end{pmatrix}.$$

It is easy to check that for the time delay $\tau = 0.8$, the corresponding system without impulses is unstable. The numerical simulation of this retarded dynamical system with respect to initial functions;

$$\phi_1(t) = \begin{cases} 0, & t \in [-0.8, 0), \\ 2.8, & t = 0; \end{cases} \qquad \phi_2(t) = \begin{cases} 0, & t \in [-0.8, 0), \\ -1.4, & t = 0; \end{cases} \qquad \phi_3(t) = \begin{cases} 0, & t \in [-0.8, 0), \\ 2.1, & t = 0. \end{cases}$$

is given in Fig. 1.

It is easy to see that $\lambda_{\max}(A^T + A + E) = 1.8819$, $\|B\|^2 = \lambda_{\max}(BB^T) = 0.0844$ and $\|E + C_k\| = 0.6$. By taking $q = 25$, $\gamma = 5$, $\lambda = 2$, $\sigma = 6.2$ and $t_{k+1} - t_k = 0.06$, it is easy to verify that all the conditions of Corollary 2.1 are satisfied:

(i) $q = 25 \ge \gamma e^{\lambda\tau} = 24.7652$, $\lambda_{\max}(A^T + A + E) + q\|B\|^2 = 3.9919 \le \sigma - \lambda = 4.2$;
(ii) $\ln \|E + C_k\| = -0.5108 < -(\sigma + \lambda)(t_{k+1} - t_k) = -0.4920$;

which means the impulsive retarded dynamical system (19) is globally exponentially stable with convergence rate 1. This conclusion cannot be derived by applying the corresponding exponential stability results for impulsive retarded differential equations given in the literature [1,2], since the length of the impulsive intervals is excessively less than the time delays, i.e., $t_k - t_{k-1} = 0.06 < \tau = 0.8$. Fig. 2 illustrates the change process of the state variables of the delay system (19) in the time interval [0, 1.4].

Remark 2.2. Obviously, it can be seen that impulses do contribute to the global exponential stability of the retarded differential system even if the corresponding system without impulses is unstable. It should be mentioned that our results allow us to develop an effective impulse control strategy to stabilize an underlying retarded dynamical system. And it is particularly meaningful for some practical applications.

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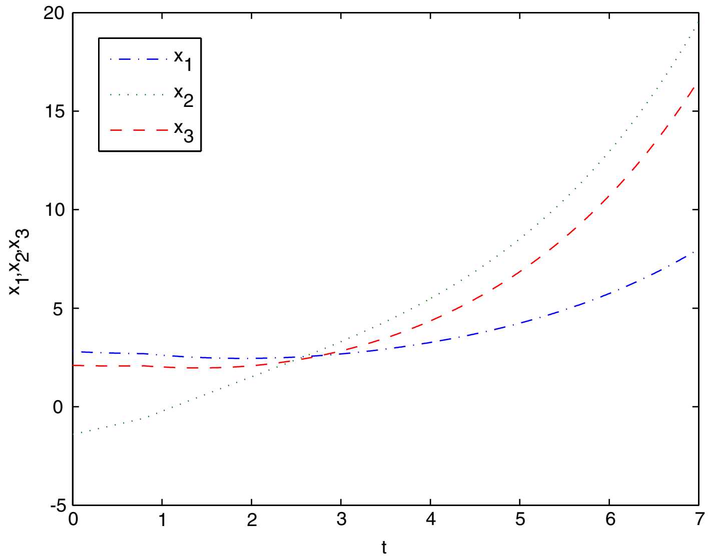

Fig. 1. System without impulses.

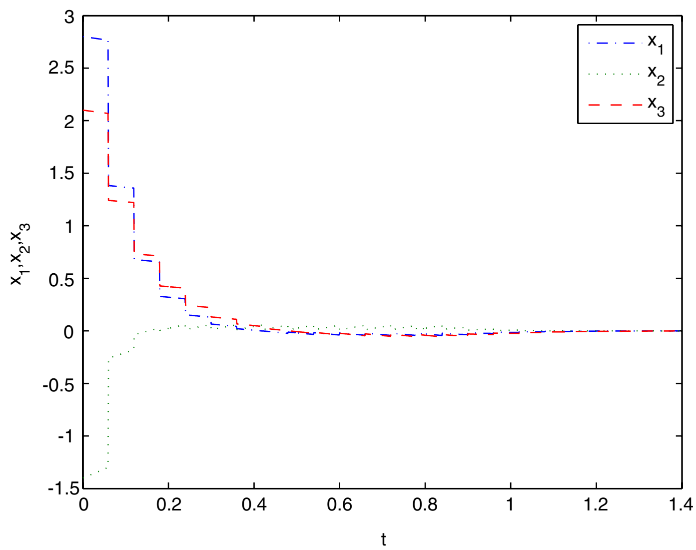

Fig. 2. Impulsively stabilized system.

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